Production planning plays a central role in the successful management of any production-oriented company. Production planning is typically multiobjective in nature and the management thereof generally consists of balancing/resolving many conflicting objectives. Previous works have shown that successful production planning can be achieved using multiobjective optimization. In this paper, a production-planning model conducive to optimization is developed and used with the preference-based optimization method Linear Physical Programming (LPP). Machine yield rates and production time are important components of the proposed model and examples that illustrate the optimization process. The key contribution of this work is in the application of LPP to a newly developed production planning model. The benefit of LPP is that it capitalizes on latent human experience and previous design knowledge when such is available. Otherwise, LPP effectively helps the designer explore the design space.

Keywords: Production planning; Physical programming; Goal programming; Multiobjective optimization

NOMENCLATURE

- $P_k$ slack variable for the total number of Product $k$’s produced (Eq. 20a)
- $t_\beta$ slack variable for the total production time (Eq. 20a)
- $m\tilde{C}$ matrix whose generic element is $m\tilde{c}_j$
- $pp\tilde{C}$ matrix whose generic element is $pp\tilde{c}_i$
- $c$ total cost to produce all products (Eq. 9)
- $m\tilde{c}_j$ raw material cost per unit mass of Material $j$ (Eq. 8)
- $pp\tilde{c}_i$ production cost per part for Machine $i$ (Eq. 8)
- $D$ matrix whose generic element is $d_{qi}$
- $d_{qi}$ defect rate for Machine $i$ when producing Part $q$ (Eq. 1)

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\( ^t\mathbf{\hat{M}} \) matrix whose generic element is \( ^t\hat{m}_{ij} \)

\( ^a\mathbf{\hat{M}}^q \) matrix whose generic element is \( ^a\hat{m}_{ij}^q \)

\( ^w\mathbf{\hat{M}}^q \) matrix whose generic element is \( ^w\hat{m}_{ij}^q \)

\( ^r\hat{m}_{ij}^q \) amount of Material \( j \) required to produce all Part \( q \)'s using Machine \( i \) (Eq. 2)

\( ^r\hat{m}_{ij} \) amount of Material \( j \) required per Part \( q \) (Eq. 2)

\( ^a\hat{m}_{ij}^q \) fraction of Material \( j \) used when producing Part \( q \), using Machine \( i \) (Eq. 2)

\( ^w\hat{m}_{ij} \) total amount of wasted Material \( j \) (Eq. 3)

\( ^w\hat{m}_{ij}^q \) fraction of Material \( j \) wasted when producing Part \( q \), using Machine \( i \) (Eq. 3)

\( ^p\mathbf{N} \) matrix whose generic element is \( ^p n_{ij} \)

\( ^pr\mathbf{N} \) matrix whose generic element is \( ^pr n_{ik} \)

\( ^{ap}n_{ik} \) number of Part \( q \)'s available for the assembly of Product \( k \) (Eq. 4)

\( ^{dp}n_{qk} \) total number of defective Part \( q \)’s (Eq. 1)

\( ^m\mathbf{N} \) number of distinct machine types (Eq. 1)

\( ^m\mathbf{N} \) number of distinct materials (Eq. 8)

\( ^p\mathbf{N} \) total number of products produced (Eq. 7)

\( ^p n_k \) total number of Product \( k \)'s produced (Eq. 7)

\( ^r n_{ij}^q \) total number of Part \( q \)'s produced using Machine \( i \) (Eq. 1)

\( ^{apr}n_{ik} \) number of Part \( q \)'s required to assemble one unit of Product \( k \) (Eq. 4)

\( ^p\mathbf{N} \) number of distinct product types (Eq. 6)

\( ^pt\mathbf{N} \) number of distinct part types (Eq. 3)

\( p \) total profit from the production of \( ^p\mathbf{N} \) products (Eq. 12)

\( r \) total sales revenue (Eq. 11)

\( ^{dp}r \) total revenue from defective-part sales (Eq. 10)

\( ^p r \) total revenue from product sales (Eq. 10)

\( ^{wm}r \) total revenue from wasted-material sales (Eq. 10)

\( ^{dp}\mathbf{\hat{S}} \) matrix whose generic element is \( ^{dp}\hat{s}_{q} \)

\( ^p \mathbf{\hat{S}} \) matrix whose generic element is \( ^p\hat{s}_{k} \)

\( ^{wm} \mathbf{\hat{S}} \) matrix whose generic element is \( ^{wm}\hat{s}_{j} \)

\( ^{dp}s_{q} \) selling price per unit of defective Part \( q \) (Eq. 10)

\( ^p s_{k} \) selling price per unit of Product \( k \) (Eq. 10)

\( ^{wm}s_{j} \) selling price per unit of wasted Material \( j \) (Eq. 10)

\( T \) matrix whose generic element is \( t_{ij} \)

\( t \) total production time to produce \( ^p\mathbf{N} \) products (Eq. 13)
1 INTRODUCTION

Good production planning is considered essential to successful production-based businesses and to good management. Production planning is generally multiobjective in nature and involves conflicting objectives, such as those associated with the cost of holding inventory and the benefits that come from doing such. Production planning managers are required to find a suitable balance between all of the production needs and objectives. Typical goals of any production planning process are to find the optimal amount of parts to produce, the optimal amount of time needed to produce them, and maximize profit while doing so. Production planning, since it is multiobjective, can benefit greatly from advances in multiobjective optimization. One such advance is the development of Physical Programming (PP); a method for optimization that effectively allows for expressing Decision Maker (DM) preferences and design knowledge. In this paper we use Linear Physical Programming (LPP) to solve a proposed production-planning model. The examples given illustrate the use of LPP and the proposed production model.

1.1 Survey of Previous Work

Because of its significant role in business management and profitability, production planning has been the focus of significant research over the last century. More recently some of this work has employed computational optimization as a means for successful production planning. Many researchers have individually considered some of the many aspects associated with production planning. Dauzere-Peres et al. [7] solve a production-planning problem that considers the continuous arrival of raw materials. The authors call this problem a continuous-time production-planning problem. The problem is solved by finding the optimal production rates and the optimal times at which production rates are to be changed. Production cost is reduced by allowing production rates to change at times other than at the end of production periods. Linear programming is used to solve the problem.

Ching and Zhou [5] examine a failure prone manufacturing system that is subject to random breakdown and repair. A single-machine one-product-type system is considered in the study. The authors seek to find the optimal hedging point for a machine-inventory model they develop. Numerical examples are provided that show the effectiveness of the developed model. In Ref [9], Huang et al. describe two general strategies for protecting against production uncertainties and fluctuations in demand. They are (i) build inventory, and (ii) temporarily increase production capacity. The objective of their model is to find the balance between increasing inventories, increasing capacity, and decreasing backlogged

\[ t_i \] total production time for Machine \( i \) (Eq. 14)

\[ t_{qi} \] production time for one unit of Part \( q \) using Machine \( i \) (Eq. 14)

\( Y \) matrix whose generic element is \( y_{qi} \)

\( y_{qi} \) yield rate for Machine \( i \) when producing Part \( q \) (Eq. 6)
demand. They emphasize in their research that uncertainty renders production planning very difficult and highly challenging. They suggest that modeling production uncertainties and optimizing actions that can be taken to hedge against uncertainties will lead to significant success in production planning.

Perkins and Srikant [26] develop a fluid-flow model for manufacturing systems that seeks to minimize inventory and shortfall costs. In their first model, machines are considered reliable, while in their second they are considered failure prone. The process is considered to have fluid flow because high volume manufacturing is assumed. The first model is optimized using sequential quadratic programming with linear constraints. Kleijnen [13] describes a decision support system for production planning. A case study is given wherein statistical design and analysis techniques are demonstrated. The statistical methods demonstrated in his work lead to combinations of the production planning system variables that were better than those chosen by intuition. Among other things, the author’s methods seek to maximize the number of productive hours.

Lee and Plenert use linear integer programming for mixed product optimization to solve a Theory of Constraints (TOC) problem [16]. In their model, they seek to maximize profit and explore methods for determining alternative product routings that also result in a maximization of profit. The authors show that the mixed product optimization methods used result in better achievement of the goals associated with the TOC philosophy.

Production planning problems are often linear in nature, making Linear Programming (LP) a suitable optimization approach. Unfortunately, traditional Linear Programming is often viewed as too limiting in that the associated Aggregate Objective Function (AOF) consists simply of a hyperplane that offers inadequate flexibility in addressing conflicting objectives. Even when the AOF consists of the weighted sum of several objectives, this critique still holds in many respects. This realization is one of the strong motivations for employing goal programming.

Charnes and Cooper, and numerous other researchers, have made major contributions to the development of Linear Goal Programming (LGP) [1-3, 6, 10-12, 15, 17, 28]. The power of LGP lies in its use of double-sided preferences about a target value. This results in the AOF no longer being a hyperplane in objective space but instead a hyperplane in the expanded space that includes deviational variables. Therefore, the resulting problem being solved is in fact an LP formulation, albeit in a different space than the original. This realization has been a source of significant debate among researchers. We believe this debate (whether LGP is actually LP) is based on a false question, as neither answer would amplify our understanding of any relevant issue.

Because of the added flexibility achieved by using Goal Programming (GP), research that utilizes GP has been conducted to optimize the production process. Golany et al. [8] present a GP inventory control method that is applied in the context of a large chemical plant, and is used to improve the production and inventory planning process. Lee and Kwak [14] use a GP based model to solve a resource allocation problem. In both of these cases, GP is used to capitalize on the decision maker’s goals and preferences. One significant drawback to the GP method is the precarious task of establishing proper deviational weights. Lee and Kwak use the Analytic Hierarchy Process (AHP) to prioritize each goal. Sarma et al. [27] use lexicographic GP for production planning optimization. In their work, the problem is first formulated as a vector maximization that generates a number of
efficient solutions, upon which the DM expresses preference for two or more. The final solution is obtained using LGP.

Messac et al. [24] explore the use of Physical Programming (PP) [19] in production planning. An optimization-based model for production planning is proposed, and Linear Physical Programming (LPP) is used as an effective tool for addressing the conflicting nature of the problem objectives. A numerical example is provided that illustrates the flexibility of such optimization-based models, and of their proposed model in particular.

The works described above are but a few examples that show that computational optimization in production planning can bring significant benefits to production systems. In this paper, we amplify the use of physical programming in production planning. In particular, we model a generic production system and investigate the impact of the developed model, which readily lends itself to practical applications. Machine yield rate (fraction of good parts made) and machine efficiency (fraction of material not wasted) are important aspects of this model. The key contribution of this paper is the extended application of the PP method in the production planning context.

1.2 Research Motivation

In this paper we propose the use of Linear Physical Programming (LPP) as an effective way to address the production-planning problem. An in-depth discussion of the PP method is presented in Ref. [19]. Section 3 of this paper provides a synopsis of the LPP method. The use of LPP is in part motivated by some significant shortcomings of the GP method with regards to weight setting – although we regard GP as a significant improvement over traditional linear programming. It can be easily seen that the GP method does provide a more flexible means for guiding the solution process than does traditional LP. The GP method involves choosing a single target value for each objective, which can provide great advantage over traditional LP. However, this advantage comes at a severe price. Choosing the targets generally requires the designation of physically meaningless weights associated with the deviation variables. Further, these weights are difficult to correctly identify. It can be argued that this requirement significantly removes the appeal of the GP method, since the final solution heavily depends on these physically meaningless weights. In addition, the final solution also depends strongly on the single target value chosen per design metric.

The LPP method provides an approach to deal with several objectives in a way that only requires the DM to specify physically meaningful targets, and not weights. In LPP, preference is expressed for each design objective in a flexible way that involves several preference values per design metric, thereby removing undue dependence on a single target. In addition, the designer does not need to specify meaningless weights. These benefits, achieved by using PP, have been described in various publications [4, 18, 20-23, 29].

The paper is organized as follows. The production-planning model is developed in Section 2, where each of its components is presented in sequence. Section 2 also presents the optimization problem formulation of the developed model. In Section 3, a synopsis of the linear physical programming method is presented. Physical programming is presented as an ideal optimization approach for use in production planning. Examples are presented in Section 4, and concluding remarks provided in Section 5.
2 PRODUCTION PLANNING MODEL

The production-planning model developed in this section includes some of the most important issues related to production planning. In particular, we develop a model that results in the maximization of the number of products produced, and profit; and the minimization of production time, subject to material availability, machine production time and yield rates. These multiple objectives are simultaneously optimized in the proposed model.

2.1 Model Components Definition

We provide below important definitions that will facilitate the development of our model formulation and the ensuing discussions.

Production Rate and Yield for each Machine Type

This production-planning model assumes that we have \( M \) distinct machine types. Each machine type produces at a different rate, wastes different percentages of material (scrap) when producing a part type, and produces a fraction of defective parts. Both wasted-material and defective-parts may be sold. Machines may refer to actual physical machines, or also to sub-manufacturing units. There may be more than one machine of a given type.

Required Material Amount, and Part Types

Each product is composed of one or more parts, and parts are composed of one or more materials. Each part type is distinguished by the amount and type of each material it is comprised of. A Part-Materials matrix, \( \tilde{M} \), defines the amount of materials required to produce each part, and it contains the generic element \( \tilde{m}_{jq} \), which denotes the amount of Material \( j \) needed to produce one unit of Part \( q \).

We let \( m \) denote the number of different raw material types used in the production system, and \( n \) denotes the number of part types that can be produced in the production system. It is likely that some of the elements in the Part-Materials matrix will be zero. When \( \tilde{m}_{jq} = 0 \), Part \( q \) is not composed of Material \( j \).

Product Type

A product type is defined by the composition of the different parts from which it is assembled. A Product Assembly matrix, \( pr \) , contains the number of parts required to assemble one unit of a particular product. The matrix \( pr \) is defined such that its generic entry, \( pr_{qk} \), is the number of Part \( q \)'s required in the assembly of one unit of Product \( k \). In the case where \( pr_{qk} = 0 \), Product \( k \) is not composed of Part \( q \).
Yield Rate
Because production planning is subject to uncertainties, it can be important to include the yield rates for each machine. Historical data for each machine is used to predict the machine yield rates, which are used during the optimization process.

In an actual production system, the quantity of good parts produced can be estimated by multiplying the historical yield by the quantity produced. We introduce a Part Yield Rate matrix $Y$, with generic entry $y_{qi}$, which denotes the yield rate for Machine $i$ when producing Part $q$. For example, if Machine $i$ produces one defective unit of Part $q$ for every five units of Part $q$ produced, then $y_{qi} = 0.8$ (see Figure 1). Also, we let $d_{qi} = 1 - y_{qi}$ be the generic entry in the Part Defect Rate matrix $D$.

The fraction of material used by a particular machine is defined in the Material Yield Rate matrix $\hat{\mu}^q$, where $\hat{\mu}^q_{ij}$ denotes the fraction of Material $j$ used when producing Part $q$ using Machine $i$. We also define the fraction of material wasted to be $\hat{\omega}^q_{ij}$, where $\hat{\omega}^q_{ij} = 1 - \hat{\mu}^q_{ij}$. For example, Machine $i$ wastes ten percent of Material $j$ to produce each unit of Part $q$, thus $\hat{\omega}^q_{ij} = 0.9$. Machine $z$, on the other hand, wastes five percent of Material $j$ to produce each unit of Part $q$, thus $\hat{\omega}^q_{zj} = 0.95$. Historical data and/or manufacturing process specifications may be used to determine the amount of material wasted by each machine – or its effectiveness. Figure 1 illustrates the above-described yield rates.

**FIGURE 1  Graphical Description of Selected Model Components**

Total Number of Manufactured Parts
The decision variables in this model are the numbers of parts produced by each machine. These variables are denoted by the matrix $^pN$ such that its generic entry, $^p_n_{qi}$, is the number of Part $q$’s (good and defective) produced using Machine $i$.

A portion of the parts produced is defective, and cannot be used to assemble the final products. The number of defective Part $q$’s, $^d_p n_{q}$, can be evaluated as

$$^d_p n_{q} = \sum_{i=1}^{M} ^p_n_{qi} d_{qi}$$ (1)

The amount of Material $j$ needed to produce the Part $q$’s on Machine $i$ (including the wasted material) is given by $^r_m^{qj}$. More specifically, it is defined as

$$^r_m^{qj} = \frac{^p_n_{qi} ^r \hat{m}^{qj}_{ij}}{^u \hat{\omega}^q_{ij}}$$ (2)

The total amount of wasted Material $j$ is given by
\[ w m_j = \sum_{q=1}^{\hat{m}_j} \sum_{i=1}^{\hat{w}_j} m_q^i w_q^i \]  

(3)

**Total Number of Manufactured Products**

The variable \( P_k \) denotes the number of Product \( k \) ’s produced. Because Product \( k \) may require a different number of each part type, the total number of Product \( k \) ’s that can be made is dependent on its part that is least available (see Figure 2). The total number of Product \( k \) ’s produced is

\[ P_k = \min_{q} \frac{ap_{qk}}{pr_{qk}} \]  

(4)

where \( ap_{qk} \) is the number of Part \( q \) ’s available to make Product \( k \), and \( pr_{qk} \) is the number of Part \( q \) ’s required to make one unit of Product \( k \), such that \( pr_{qk} \neq 0 \). Figure 2 illustrates the relationship described in Eq. (4). That is, it illustrates the maximum number of products produced, given the required product composition and number of parts available.

**FIGURE 2   Number of Manufactured Products**

An important observation is that in order to avoid producing any excess parts, we must have, by definition, for each Product \( k \)

\[ \frac{ap_{qk}}{pr_{qk}} = \frac{ap_{ik}}{pr_{ik}} \]  

(5)

for all \( q \) and \( i \), where Parts \( q \) and \( i \) are distinct parts that comprise Product \( k \), and \( pr_{qk}, pr_{ik} \neq 0 \). Further, the total number of Part \( q \) ’s used must equal the total number of good Part \( q \) ’s produced, which is enforced by the following constraint

\[ \sum_{k=1}^{\hat{p}_n} ap_{qk} = \sum_{i=1}^{\hat{p}_n} p_{qi} y_{qi} \]  

(6)

The total number of products, \( P_n \), is the sum of all product types, and is one of the design metrics maximized in this model

\[ P_n = \sum_{k=1}^{\hat{p}_n} P_k \]  

(7)

**Cost of Manufactured Parts**

The cost incurred to produce the parts is assumed to be a linear function of the cost of raw materials used and of the number of products produced by a given machine. The raw materials cost, \( m_c \), and production cost, \( p_c \), are defined as
\[ m \mathbf{c} = \sum_{j=1}^{n} \left( \sum_{q=1}^{n} \sum_{i=1}^{n} m_{ij} \mathbf{c}_{j} \right)^{m} \]

\[ p \mathbf{c} = \sum_{i=1}^{n} \left( \sum_{q=1}^{n} p_{ni} \mathbf{c}_{i} \right)^{p} \]

where \( m \mathbf{c}_{j} \) is the cost of Material \( j \) per unit mass, and \( p \mathbf{c}_{i} \) is the operating cost per part for Machine \( i \). The total cost to produce all products, including the cost to produce defective parts, is

\[ c = m \mathbf{c} + p \mathbf{c} \]

**Production System Profit**

The total revenue from the production system comes from three sources. The first source is from the sale of products, \( P \), while the second and third sources of revenue are the sale of defective parts, \( D \), and sale of wasted materials, \( W \), respectively. Each is defined as follows

\[ P = \sum_{k=1}^{n} p_{k} S_{k}; \quad D = \sum_{q=1}^{n} d_{q} S_{q}; \quad W = \sum_{j=1}^{n} w_{j} S_{j} \]

The selling price of Product \( k \) is \( P_{k} \), the selling price of defective Part \( q \) is \( D_{q} \), and \( W_{j} \) is the selling price of wasted Material \( j \). The total revenue is then given by

\[ r = P + D + W \]

The profit is the difference between the sales revenue and the production costs, and is given by

\[ p = r - c \]

In this production-planning model one of the objectives is to maximize profit, \( p \).

**Time Required to Manufacture all Products**

The amount of time needed to produce \( P \) products is now considered. A time matrix, \( T \), has the generic element, \( t_{qi} \), which denotes the amount of time needed to make a single Part \( q \) using Machine \( i \).

A production system may operate in parallel, series, or a combination of the two. In this model we explore two possible scenarios for the production of parts: parallel and series. In a parallel production system one of each machine type is available for each part type (see Figure 3). Therefore, the total time to produce \( P \) products in parallel is given by

\[ t = \max_{q,i} t_{qi} p_{ni} \]

In contrast, only one of each machine type is available for use in a series production system. As in the parallel production case, machines of different types can also be used simultaneously. Stated differently, in parallel production different Machine \( i \)'s (for a given \( i \)) can be used in parallel to produce the same part type. In series production, only one
machine type is available for production, and must therefore produce different part types in series. The total time to produce \( P_n \) products in series is given by

\[
t = \max_i t_i; \quad t_i = \sum_{q=1}^{P_n} t_{qi} n_{qi}
\]

where \( t_i \) is the total production time of Machine \( i \). Figure 3 contrasts the parallel and series production scenarios in the case of three part types and two machine types.

**FIGURE 3 Parallel and Series Production Strategies**

Note that in order to produce good parts, defective parts will also be produced. The total time, \( t \), evaluated above includes the time taken to produce these defective parts. In this production-planning model, minimizing the production time, \( t \), is one of the objectives. In the following section, the model components defined above are used to formulate the production planning optimization problem statement.

### 2.2 Optimization Problem Statement

The general production-planning model presented and used in this research is defined by the following formulation and is supported by the preceding definitions.

\[
\text{opt}_{p, n, t} \left\{ \begin{array}{l}
P_n \\
q_k \\
t
\end{array} \right\}
\]

subject to

\[
P_n = \sum_{k=1}^{P_n} p_{nk}; \quad P_k = \min_q \sum_{i=1}^{P_i} n_{q_k} ; \quad k = 1, \ldots, P_k; \quad q = 1, \ldots, P_n
\]

\[
\sum_{k=1}^{P_n} p_{q_k} = \sum_{q=1}^{P_n} n_{q_k} q_{ki} ; \quad q = 1, \ldots, P_n
\]

\[
\frac{p_{q_k}}{p_{n_k}} = \frac{p_{q_k}}{p_{n_k}} ; \quad q, i = 1, \ldots, P_n; \quad k = 1, \ldots, P_k
\]

\[
p = r - c \quad \text{(See Eq. 12)}
\]

\[
t = \max_{q, i} t_{qi} p_{n_q}; \quad q = 1, \ldots, P_n; \quad i = 1, \ldots, M
\] (for parallel)

\[
t = \max_i t_i; \quad t_i = \sum_{q=1}^{P_n} t_{qi} n_{qi}; \quad i = 1, \ldots, M
\] (for series)

A full description of each variable in Eq. (15) is given in Section 2.1. Equation (15f) applies when parts are produced in parallel, whereas Eq. (15g) is applied in series production.
Minmax and maxmin formulations are embedded in the above optimization problem statement. Specifically, we seek to minimize the time to produce all products, \( t \), which is in itself the maximum over \( q \) and \( i \) (Eqs. 13, 14). Also, we seek to maximize the number of products sold, which is the minimum number that can be produced by the available number of composing parts (Eq. 4). Below we describe the approach used to solve the minmax problem.

**Minmax problem solution approach:**
The minmax optimization problem is transformed from

\[
\text{Problem A} \quad \min_x \max_{i} f_i(x); \quad i = 1, \ldots, r
\]

into the following equivalent optimization problem

\[
\text{Problem B} \quad \min \beta
\]

subject to

\[
f_i(x) \leq \beta; \quad i = 1, \ldots, r
\]

where

\[
\beta = \max_{i} f_i(x); \quad i = 1, \ldots, r
\]

A similar transformation can be employed for a maxmin problem. Using the above transformation, the optimization problem statement becomes

\[
\begin{align*}
\min & \quad -\sum_{k=1}^{n} \beta_k \\
\text{subject to} & \\
& -p = -\sum_{k=1}^{n} \beta_k \\
& t = \beta
\end{align*}
\]

\[
\begin{align*}
\sum_{k=1}^{n} n_{q_k} &= \sum_{i=1}^{n} n_{q_i} y_{q_i}; \quad q = 1, \ldots, p \quad n \\
\frac{ap n_{q_k}}{pr n_{q_k}} &= \frac{ap n_{k}}{pr n_{k}}; \quad k = 1, \ldots, p; \quad q, i = 1, \ldots, p \quad n \\
\frac{p c}{\beta_k} &\leq \frac{ap n_{q_k}}{pr n_{q_k}}; \quad k = 1, \ldots, p; \quad q = 1, \ldots, p \quad n \\
p &= r - c \quad \text{(See Eq. 12)} \\
\beta &\geq t_{q_i} n_{q_i}; \quad q = 1, \ldots, p; \quad i = 1, \ldots, M \quad n
\end{align*}
\]

For the case when production takes place in series, Eq. (20f) is replaced by Eq. (20g).
LINEAR PHYSICAL PROGRAMMING

Numerous published works show physical programming to be an effective method for use in engineering and business applications [4, 18, 20, 22, 23, 29]. We use Linear Physical Programming (LPP) to solve the production-planning model developed in Section 2. Section 3 provides a brief description of the method. Additional information regarding the Physical Programming method can be found in Refs. [19, 25].

Within the physical programming procedure, the Decision Maker (DM) expresses his or her preferences with respect to each criterion using four different classes by declaring that each belong to one of the classes. Each class comprises two cases, hard and soft, referring to the sharpness of the preference. Figure 4 depicts the qualitative and quantitative meanings of each soft class. The value of the criterion under consideration, \( g_p \), is shown on the horizontal axis, and the function that will be minimized for that criterion, \( Z_p \), hereby called the class-function, is shown on the vertical axis. All soft class functions will become constituent components of the aggregate objective function.

**FIGURE 4** Class Functions for Linear Physical Programming

Physical programming allows the user to express preferences with regard to each criterion with more specificity and flexibility than by simply saying minimize, maximize, greater than, less than, or equal to. The preferences are characterized by degrees of desirability as seen in Figure 4. Consider, for example, the case of Class 1S. The preference ranges are:

- **Ideal** range: \( g_p \leq t_{p1}^+ \)
- **Desirable** range: \( t_{p1}^+ \leq g_p \leq t_{p2}^+ \)
- **Tolerable** range: \( t_{p2}^+ \leq g_p \leq t_{p3}^+ \)
- **Undesirable** range: \( t_{p3}^+ \leq g_p \leq t_{p4}^+ \)
- **Highly Undesirable** range: \( t_{p4}^+ \leq g_p \leq t_{p5}^+ \)
- **Unacceptable** range: \( g_p > t_{p5}^+ \)

The parameters \( t_{p1}^+ \) through \( t_{p5}^+ \) are physically meaningful constants that express the DM’s preference associated with the \( i \)th generic design metric.

Class-functions are used to map design metrics into non-dimensional, strictly positive, real numbers. This mapping, in effect, transforms design metrics with disparate units and physical meaning onto a dimensionless scale through a unimodal function. Consider the
first curve of Figure 4. When the value of the criterion, $g_p$, is less than $t_{p1}^+$ (ideal range), the value of the class function is small, and requires little further minimization. When, on the other hand, the value of the metric, $g_p$, is between $t_{p4}^+$ and $t_{p5}^+$ (highly-undesirable range), the value of the class function is large, and necessitates significant minimization. Stated simply, the value of the class-function for each design metric governs the optimization path in objective space. Among the most important desired properties of the class function are that (i) they are nonnegative, continuous, piecewise linear, and convex, and (ii) the value of the class function, $Z_p$, at a given target level (say $t_{p1}^+$) is the same for all class types.

Based on the above stated properties, linear physical programming determines the weights ($\tilde{w}_{ps}^+$ and $\tilde{w}_{ps}^-$, see Eq. (21a)) that represent the incremental slope of the class (preference) functions, $Z_p$. Thus, $Z_p$ can be expressed as a piecewise linear function of criterion $g_p$. Reference [25] presents the algorithm for determining the weights. The aggregate objective function (to be minimized) is then constructed as a weighted sum of deviations over all ranges ($s = 2,...,5$) and criteria ($p = 1,...,P$). The resulting LPP formulation is as follows:

$$\min_{d_{ps}, d_{ps}^+, x} J = \sum_{p=1}^{P} \sum_{s=2}^{5} \left( \tilde{w}_{ps}^+ d_{ps}^+ + \tilde{w}_{ps}^- d_{ps}^- \right)$$

subject to

$$g_p(x) - d_{ps}^+ \leq t_{p,s-1}^+; \quad d_{ps}^+ \geq 0; \quad g_p(x) \leq t_{p5}^+$$  \hspace{1cm} (21b)

(for classes 1S, 3S, 4S; $p = 1,...,P$; $s = 2,...,5$)

$$g_p(x) + d_{ps}^- \geq t_{p,s-1}^-; \quad d_{ps}^- \geq 0; \quad g_p(x) \geq t_{p5}^-$$  \hspace{1cm} (21c)

(for classes 2S, 3S, 4S; $p = 1,...,P$; $s = 2,...,5$)

$$x_{\min} \leq x \leq x_{\max}$$  \hspace{1cm} (21d)

where $d_{ps}^-$ and $d_{ps}^+$ respectively denote the negative and positive deviations of the objective (criterion) value $g_p(x)$ from target levels $t_{p,s-1}^-$ and $t_{p,s-1}^+$; and $g_p(x)$ is a linear function of $x$. The first constraint (Eq. 21b) applies to criteria belonging to all classes except Class 2S, while the second constraint (Eq. 21c) applies to criteria belonging to all classes except Class 1S. Finally, Eq. (21d) shows the side constraints for the decision vector $x$.

In the next section, the LPP method is used to optimize the production planning model given in Section 2.

4  EXAMPLES

In this section we present two examples that demonstrate the usefulness of the Linear Physical Programming (LPP) method for solving the production-planning model developed
in Section 2. The first example is used to compare parallel and series production approaches. In the second example, different priorities are placed upon the three design metrics, \( p_n, p_t, \) and the optimization results are compared. For both examples, the following system description applies:

(i) The number of distinct types of machines in the system is two, \( (M_n = 2) \)
(ii) The number of distinct raw material types used is three, \( (m_n = 3) \)
(iii) The number of distinct part types produced by the system is three, \( (p_t n = 3) \)
(iv) The number of distinct product types produced by the system is three, \( (P_t n = 3) \)

Additionally, the constants defined in Section 2 are given as

\[
\begin{align*}
\mathbf{r} \tilde{M} &= \begin{bmatrix} 2.25 & 3.8 & 0 \\ 0 & 4.1 & 3.5 \\ 1.8 & 0 & 3.5 \end{bmatrix}, \quad \mathbf{p} \tilde{N} &= \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 6 \\ 1 & 4 & 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0.95 & 0.87 \\ 1 & 0.92 \end{bmatrix}, \\
\mathbf{u} \tilde{M}^l &= \begin{bmatrix} 0.97 & 0.95 & 0.87 \\ 0.89 & 0.85 & 0.92 \end{bmatrix}, \quad \mathbf{u} \tilde{M}^r &= \begin{bmatrix} 0.9 & 0.98 & 1 \\ 0.88 & 0.85 & 0.79 \end{bmatrix}, \quad \mathbf{u} \tilde{M}^* = \begin{bmatrix} 0.82 & 0.95 & 0.93 \\ 0.8 & 0.97 & 0.93 \end{bmatrix}, \\
\mathbf{T} &= \begin{bmatrix} 0.5 & 1 \\ 0.8 & 0.4 \\ 0.2 & 0.1 \end{bmatrix}, \quad m \tilde{C} = \begin{bmatrix} 0.05 & 0.08 & 0.1 \end{bmatrix}, \quad pp \tilde{C} = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}, \\
p \tilde{S} = [6 \ 12 \ 14], \quad dp \tilde{S} = [0.55 \ 0.7 \ 0.55], \quad wm \tilde{S} = [0.02 \ 0.04 \ 0.02]
\end{align*}
\]

### 4.1 Example 1: Parallel and Series Production Approaches

In Example 1, the production planning model developed in Section 2 is used to obtain the number of parts to be produced on each machine, and the resulting values of the objectives. Both the parallel and series production approaches are considered, and the results are compared. The decision maker preferences used in Example 1 are given in Table I.

#### TABLE I Preference Table for Examples 1 and 2

The parallel and series production approaches were formulated according to the production planning model described in Section 2. The optimized results, solved using the LPP method, are shown in Tables II and III. The optimal values for the number of parts produced and number of products produced have been respectively rounded up and down to the nearest integer in order to maintain feasibility.

#### TABLE II Optimized Design Variables for Example 1

#### TABLE III Optimized Design Metrics for Example 1

The results in Table II show that Machine 2 is not used to produce Part 1 or Part 3. This is a reflection of the part and material yield of Machine 2, which is less than that of
Machine 1, combined with the higher operational cost of using Machine 2. Machine 2 is used to produce Part 2 because of the significantly lower processing time it takes to produce each unit of Part 2, as compared to the processing time Machine 1 requires.

Examining the DM specified preferences given in Table I, we can see that in the parallel case the resulting design metric values lie in the desirable range for all three design metrics. This indicates that if the number of parts given in Table II is assigned to each machine, the production system will operate at a level that is desirable to the DM. In the series case, the number of products produced and profit lie in the undesirable range, while the production time lies in the tolerable range.

It is observed that when the same DM-specified preferences are used in both the parallel and series production system, the resulting design metric values reflect the superiority of using the parallel production system — for this set of DM preferences. Figure 5 illustrates the difference in the two production systems. The total time needed for series production is higher than that of the parallel production system, and the profit and productivity are lower for the series production.

**FIGURE 5 Parallel and Series Production Comparison – Example 1**

Although the series production system results in less desirable values for all three design metrics, there may be times when it is still necessary to use a series production system. When producing in series, there may be less supervision required over the production process. The series production process is less complicated compared to a parallel production system, where many operations may be occurring at the same time. The two cases exemplify extreme production scenarios. In the actual production system, any combination of the two cases can be used. Thus, knowledge about the trade-offs between parallel and series production systems, as is seen in Figure 5, gives valuable insight into the parameters used to define the production system.

### 4.2 Example 2: Emphasis on Individual Design Metrics

In Example 2, different emphasis is placed on the design metrics, $P_n$, $p$, $t$, the production planning model is optimized, and the optimization results are compared. Different emphasis on the design metrics is reflected in the designer preferences in Table I, under Example 2. In this example, only the parallel production system is used.

#### 4.2.1 Description of Three Systems

The following is a brief description of the three production systems used in this example.

**Example 2a: Highly Productive System**

In this example, increased emphasis is placed on the number of products produced. For design metric $P_n$, the preference values, listed in Table I for Example 2a, have increased compared to Example 1.
Example 2b: Highly Profitable System
In this example, increased emphasis is placed on the system profit. For design metric \( p \), the preference values, listed in Table I for Example 2b, have increased compared to Example 1.

Example 2c: Time Limited System
In this example, increased emphasis is placed on the production time. For design metric \( t \), the preference values, listed in Table I for Example 2c, have increased compared to Example 1.

4.2.2 Results and Discussion
Each production system described in Section 4.2.1 is optimized to obtain the optimal part-machine mix and the resulting design metric values. The results are given in Tables IV and V.

TABLE IV Optimized Design Variables for Example 2

TABLE V Optimized Design Metrics for Example 2

For the highly productive system, the number of products produced increased, with some adverse effects on profit and production time. The number of products produced lies in the tolerable range due to the increased value of preferences set by the DM, while profit and production time remain in the desirable range.

The optimized results for the highly profitable production system show that profit increases, which corresponds to the higher preference range specified (see Table I). Profit now lies in the tolerable range. The number of products produced remains in the desirable range, whereas the total production time degraded slightly in performance into the tolerable range.

The result of emphasizing production time preferences is that the total production time decreased, and it now lies in the tolerable range. The total number of products produced still lies in the tolerable range, while the total profit lies in the desirable range.

Figure 6 illustrates the optimization results for the three production systems, which were achieved by modifying the DM preferences. All three cases highlight the flexibility of LPP in adapting to the different preferences a DM may have. It also shows the ease of using optimization in cases where the production system changes, and shows the different operating conditions the system should employ with different DM preferences.

FIGURE 6 Emphasis on Individual Design Metric – Example 2
5 CONCLUDING REMARKS

In this paper, a production planning model is developed that takes into account machine yield rate and efficiency, and that is conducive to optimization. The model is used in examples where parallel and series production systems are considered, and where the priorities of the production objectives change. In each case the production planning model proves useful in arriving at the optimal part-machine mix needed to maximize products produced and profit, while minimizing total production time. Linear Physical Programming explicitly uses decision maker preferences, and identifies the optimal production-planning strategy. The use of physically meaningful decision-maker preferences in the optimization process provides the basis for a more meaningful optimization by those charged with the task of optimizing production systems. This is in contrast with using conventional Linear Programming with physically meaningless weights, which can pose serious difficulty to the optimization application process.

ACKNOWLEDGMENT

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FIGURE 1  Graphical Description of Selected Model Components
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<td>5x Part-2</td>
<td>20x Part-2 → 4x Product-1 → 3x Product-1</td>
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<td>8x Part-3</td>
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**FIGURE 2** Number of Manufactured Products
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P = Part Type  M = Machine Type

FIGURE 3  Parallel and Series Production Strategies
FIGURE 4 Class Functions for Linear Physical Programming
FIGURE 5  Parallel and Series Production Comparison – Example 1
FIGURE 6  Emphasis on Individual Design Metric – Example 2
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HU = highly undesirable, U = undesirable, T = tolerable, D = desirable, I = ideal
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